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ON A THEOREM OF BERKOVICH

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ABSTRACT

In a recent paper, Berkovich studied how to describe the nilpotent residual of a group in terms of the nilpotent residuals of some of its subgroups. That study required the knowledge of the structure of the minimal non-nilpotent groups, also called Schmidt groups. The major aim of this paper is to show that this description could be obtained as a consequence of a more complete property, giving birth to some interesting generalizations. This purpose naturally led us to the study of a family of subgroup-closed saturated formations of nilpotent type. An innovative approach to these classes is provided.

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1. Introduction and basic definitions

All groups considered in this paper will be finite.

In [2], Berkovich obtained some remarkable results which provide a detailed portrait of the nilpotent residual of a group in terms of the nilpotent residual of some of its subgroups.

A group is said to be **minimal non-nilpotent** if it is not nilpotent but all its proper subgroups are nilpotent. These groups were studied by Schmidt. It is known that any minimal non-nilpotent group G satisfies:

1. $\pi(G) = \{p, q\}$ and $G = PQ$ where P is a Sylow p-subgroup of G and $G' = Q$ is a Sylow q-subgroup of G.
2. P is cyclic and $|P : P \cap Z(G)| = p$.
3. $Q/Q \cap Z(G)$ is a minimal normal subgroup of $Q/Q \cap Z(G)$ and Q is special.

It is easy to check that a group possessing these three properties is minimal non-nilpotent. Following Berkovich ([2]), we call a group with this structure an $S(p,q)$-**group**.

Finally, a group G is said to be a $B(p,q)$-**group** if $G/\Phi(G)$ is an $S(p,q)$-group for primes p and q.

The main result contained in [2] is the following:

THEOREM 1 ([2, Theorem]): *Let q be a prime. Denote by $G^{\mathcal{N}}$ and by $\mathcal{X}_q(G)$ the nilpotent and q-nilpotent residuals, respectively, of a group G. The following properties hold for every group G:*

1. $\mathcal{X}_q(G) = \langle A^{\mathcal{N}} \colon A \leq G; A \in B(p,q) \text{ for } p \in \pi(G) \setminus \{q\}\rangle$,
2. $G^{\mathcal{N}} = \langle A^{\mathcal{N}} \colon A \leq G; A \in B(p,q) \text{ for } p, q \in \pi(G)\rangle$.

In other words, the second part of the theorem can be read as

$$G^{\mathcal{N}} = \langle A^{\mathcal{N}} \colon A \leq G; A \in \mathcal{B}_{\mathcal{N}}\rangle$$

where $\mathcal{B}_{\mathcal{N}}$ denotes the set of subgroups A of G such that $A/\Phi(A)$ is minimal non-nilpotent.

The main purpose of the present paper is to obtain a natural generalization of this result to more general classes. In order to proceed to that target, we shall use the following notation and terminology.

A **formation** is a class $\mathcal{F}$ of groups satisfying the following two conditions:

F1. Every homomorphic image of an $\mathcal{F}$-group is an $\mathcal{F}$-group.

F2. If G/M and G/N are both $\mathcal{F}$-groups, for normal subgroups M and N of G, then $G/M \cap N$ belongs to $\mathcal{F}$ as well.

We shall say that a formation $\mathcal{F}$ is **saturated** if the group G belongs to $\mathcal{F}$ provided that the Frattini factor group $G/\Phi(G)$ is in $\mathcal{F}$.

Given a subgroup-closed formation $\mathcal{F}$, a group G is said to be **$\mathcal{F}$-critical** if $G \notin \mathcal{F}$ but all proper subgroups of G are in $\mathcal{F}$. We shall denote the set of all $\mathcal{F}$-critical groups by $\mathrm{Crit}_S(\mathcal{F})$. It is clear that a group $G \notin \mathcal{F}$ if and only if there exists a subgroup H of G such that H is $\mathcal{F}$-critical.

Moreover, for $\mathcal{F}$ a formation and G any finite group, there exists a minimal element in the set of normal subgroups N of G such that $G/N \in \mathcal{F}$. This element is denoted by $G^{\mathcal{F}}$ and is called the **$\mathcal{F}$-residual of** G.

Associated to any subgroup-closed formation $\mathcal{F}$, we define the class $\mathcal{B}_{\mathcal{F}}$ of all groups G such that $G/\Phi(G)$ is an $\mathcal{F}$-critical group. This definition generalizes Berkovich's one to an arbitrary subgroup-closed formation.

We shall show in this paper that Berkovich's Theorem does not hold only for the class of nilpotent groups. On the contrary, it can be generalized to any subgroup-closed saturated formation, as we show in our Theorem A. Its proof is not an empty exercise of generalization. We have elaborated an alternative proof which leads to a succint proof of Berkovich's Theorem.

Following this result, the interest in $\mathcal{F}$-critical groups for different formations $\mathcal{F}$ naturally arises. For several formations $\mathcal{F}$, the structure of $\mathcal{F}$-critical groups has been widely studied throughout the last decades. For instance, the structure of $\mathcal{N}$-critical groups, where $\mathcal{N}$ denotes the class of nilpotent groups, is well known, as we have mentioned before. They were described by Schmidt (see [5; III, 5.2]). Doerk studied in [3] the $\mathcal{U}$-critical groups, where $\mathcal{U}$ is the class of supersoluble groups.

More generally, some information about $\mathcal{F}$-critical groups for an arbitrary saturated formation $\mathcal{F}$ can be found in [4; VII, 6.18].

Given a set of primes π, we shall denote by $\mathcal{S}_\pi$ the class of soluble π-groups.

Any function $f\colon \mathbb{P} \longrightarrow \{\text{formations}\}$ is called a **formation function**. Given a formation function f, we define the set $LF(f)$ as the class of all finite groups satisfying the following condition:

> $G \in LF(f)$ if for all chief factors H/K of G and for all primes p dividing $|H/K|$, we have $\mathrm{Aut}_G(H/K) \in f(p)$.

A class of finite groups $\mathcal{F}$ is called a **local formation** if there exists a formation function f such that $\mathcal{F} = LF(f)$. In such a case, we say that $\mathcal{F}$ is **locally defined** by f. By a theorem of Gaschütz, Lubeseder and Schmid [4; IV, 4.6], we know that a formation of finite groups is local if and only if it is saturated.

If f is a formation function, and we write $\mathcal{F} = LF(f)$, then f is called **inte-**

grated if $f(p) \subseteq \mathcal{F}$ for all $p \in \mathbb{P}$. In addition, f is called **full** if $f(p) = \mathcal{S}_p f(p)$ for all $p \in \mathbb{P}$.

It is well known (see [4; IV, 3.7]) that given a local formation $\mathcal{F}$, there exists precisely one full and integrated formation function F such that $\mathcal{F} = LF(F)$. Such formation function F is called the **canonical local definition** of $\mathcal{F}$.

Finally, recall that the characteristic of a class $\mathcal{F}$ of groups is defined to be the set of primes p such that the cyclic group of order p belongs to $\mathcal{F}$. Such a set is often denoted by $\operatorname{char}\mathcal{F}$.

The interest in describing all saturated formations $\mathcal{F}$ of finite groups such that every $\mathcal{F}$-critical group is either a $\mathcal{N}$-critical group or a cyclic group of prime order arose after a question of Shemetkov in the Kourovka Notebook [6, p. 84].

In [1], the second author and Pérez-Ramos provided the exact description of the subgroup-closed saturated formations of soluble groups for which the above property holds. In the soluble case, the following theorem summarizes their results.

THEOREM 2 ([1, Theorem 4]): *Let $\mathcal{F} = LF(F)$ be a subgroup-closed saturated formation of soluble groups, where F is the canonical local definition of $\mathcal{F}$. Denote $\pi = \operatorname{char}\mathcal{F}$. The following statements are equivalent:*

1. *Every soluble group in $\operatorname{Crit}_S(\mathcal{F})$ is a Schmidt group or a cyclic group of prime order.*
2. *For each prime $p \in \pi$, we have that $F(p) = \mathcal{S}_{\pi(p)} \cap \mathcal{F}$, where $\pi(p) = \operatorname{char} F(p)$.*
3. *For each prime $p \in \pi$, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathcal{F}$ is locally defined by the formation function f given by $f(p) = \mathcal{S}_{\pi(p)}$ if $p \in \pi$, and $f(q) = \emptyset$ if $q \notin \pi$.*

Our generalization of Berkovich's result naturally led us to obtain a new characterization of the formations of soluble groups satisfying the statements of the above Theorem. This effort, which partly justifies the following pages, was successfully completed in Theorems B and C.

2. Main results

The following two elementary lemmas are necessary to prepare the discussion of our main theorems.

LEMMA 1: *Let $\mathcal{F}$ be a saturated formation. If a group G is $\mathcal{F}$-critical, then so is $G/\Phi(G)$.*

Proof: Since $\mathcal{F}$ is saturated, clearly $G/\Phi(G)$ does not belong to $\mathcal{F}$. Moreover, recall that every proper subgroup of G belongs to $\mathcal{F}$ and every epimorphic image of an $\mathcal{F}$-group keeps lying in $\mathcal{F}$. It follows that every proper subgroup of $G/\Phi(G)$ lies in $\mathcal{F}$. Consequently, $G/\Phi(G)$ is $\mathcal{F}$-critical.

LEMMA 2 (see [4; VII, 6.18]): *Let $\mathcal{F}$ be a formation containing the class $\mathcal{N}$ of nilpotent groups. If G is a Schmidt group with $\Phi(G) = 1$, then $G^{\mathcal{F}}$ is an elementary abelian normal subgroup of G.*

Proof: We can assume that G has a normal Sylow p-subgroup P and a cyclic Sylow q-subgroup Q such that $G = PQ$. Clearly $\Phi(P) \leq \Phi(G) = 1$ and hence P is an elementary abelian group. The result holds since $G^{\mathcal{F}} \leq G^{\mathcal{N}} \leq P$.

We present now the extension of Theorem 1.

THEOREM A: *Let $\mathcal{F}$ be a subgroup-closed saturated formation, and let G be any group. Then*

$$G^{\mathcal{F}} = \langle A^{\mathcal{F}} \colon A \leq G; A \in \mathcal{B}_{\mathcal{F}} \rangle.$$

Proof: Let us set $T = \langle A^{\mathcal{F}} \colon A \in \mathcal{B}_{\mathcal{F}} \rangle$. Clearly $A^{\mathcal{F}} \leq G^{\mathcal{F}}$ for every subgroup A of G because $\mathcal{F}$ is subgroup-closed and therefore T is contained in $G^{\mathcal{F}}$.

Consequently, we shall be done if we prove that $G^{\mathcal{F}}$ is contained in T. In other words, we must see that G/T belongs to $\mathcal{F}$. Assume that $G/T \notin \mathcal{F}$. Then G/T has an $\mathcal{F}$-critical subgroup, A/T say. Choose now a minimal supplement A_0 of T in A. Then $A_0 \cap T$ is contained in $\Phi(A_0)$.

Since A/T is isomorphic to $A_0/A_0 \cap T$, it follows that $A_0/A_0 \cap T$ is $\mathcal{F}$-critical. It is clear then that the factor group $(A_0/A_0 \cap T)/\Phi(A_0/A_0 \cap T)$ is $\mathcal{F}$-critical as well by Lemma 1.

Consequently $A_0 \in \mathcal{B}_{\mathcal{F}}$. Therefore $A_0^{\mathcal{F}} \leq T$, and hence $A_0^{\mathcal{F}} \leq A_0 \cap T \leq \Phi(A_0)$. It follows that $A_0/\Phi(A_0) \in \mathcal{F}$. Now since $\mathcal{F}$ is saturated, we conclude that $A_0 \in \mathcal{F}$, the final contradiction. ∎

Note that choosing $\mathcal{F}$ to be the formation of q-nilpotent groups for a prime q or the class of nilpotent groups in the above theorem, one can obtain Theorem 1 as a natural consequence.

THEOREM B: *Let $\mathcal{F}$ be a subgroup-closed saturated formation of soluble groups containing the class $\mathcal{N}$ of nilpotent groups, such that every soluble group in $\mathrm{Crit}_S(\mathcal{F})$ is a Schmidt group.*

If A is a group in $\mathcal{B}_{\mathcal{F}}$, then $A^{\mathcal{N}} = A^{\mathcal{F}}$.

Proof: We shall argue by induction on $|A|$. Firstly, if $\Phi(A) = 1$, then A is an $\mathcal{F}$-critical group and consequently a Schmidt group. Hence there exists a normal abelian Sylow p-subgroup of A, P say, for some prime p. It is not difficult to see that in such a case P coincides with both the nilpotent residual and the $\mathcal{F}$-residual of A. Hence we can assume that $\Phi(A) \neq 1$. Let N be a minimal normal subgroup of A contained in $\Phi(A)$. It is clear that $A/N \in \mathcal{B}_{\mathcal{F}}$. Hence, by induction, we have that $(A/N)^{\mathcal{F}} = (A/N)^{\mathcal{N}}$. This yields $A^{\mathcal{N}}N = A^{\mathcal{F}}N$.

If $N \cap A^{\mathcal{N}} = 1$, then $A^{\mathcal{N}} = A^{\mathcal{N}} \cap A^{\mathcal{F}}N = A^{\mathcal{F}}(A^{\mathcal{N}} \cap N) = A^{\mathcal{F}}$ and we are done. Hence we can assume that $N \cap A^{\mathcal{N}} = N$; that is, N is contained in $A^{\mathcal{N}}$. This implies that $A^{\mathcal{N}} = A^{\mathcal{F}}N$. If $N \leq A^{\mathcal{F}}$, then $A^{\mathcal{N}} = A^{\mathcal{F}}$ and the theorem is true. Consequently we shall assume that $N \cap A^{\mathcal{F}} = 1$, and hence $\Phi(A) \cap A^{\mathcal{F}} = 1$.

Recall that $A/\Phi(A)$ is an extension of a p-group by a q-group for some primes p and q. Since this class is a saturated formation, we have that A is also an extension of a p-group by a q-group. But P is the unique Sylow p-subgroup of A, and thus it is clear that A/P is nilpotent. Hence $A^{\mathcal{N}}$ is a p-group, and consequently N is a p-group, too.

Let us have a look now at the structure of the $\mathcal{F}$-group $A/A^{\mathcal{F}}$. Given a subgroup H of A, denote by $\bar{H}$ the corresponding subgroup $HA^{\mathcal{F}}/A^{\mathcal{F}}$ of $A/A^{\mathcal{F}}$. Applying Theorem 2, we have that the class $\mathcal{F}$ is locally defined by a formation function f given by $f(r) = \mathcal{S}_{\pi(r)}$ if $r \in \operatorname{char}\mathcal{F}$, and $f(s) = \emptyset$ if $s \notin \pi$. Now note that $\bar{N} = NA^{\mathcal{F}}/A^{\mathcal{F}}$ is a minimal normal subgroup of $\bar{A} = A/A^{\mathcal{F}}$. Therefore,

$$\bar{A}/C_{\bar{A}}(\bar{N}) \in \mathcal{S}_{\pi(r)}$$

where $r \in \pi(|N|) = \{p\}$. That is,

$$\bar{A}/C_{\bar{A}}(\bar{N}) \in \mathcal{S}_{\pi(p)}.$$

We can conclude that

$$A/C_A(N) \cong (A/A^{\mathcal{F}})/(C_A(N)/A^{\mathcal{F}}) = \bar{A}/C_{\bar{A}}(\bar{N}) \in \mathcal{S}_{\pi(p)}.$$

If $q \in \pi(p)$, then $A \in f(p)$, which is impossible since A is $\mathcal{F}$-critical.

Consequently $q \notin \pi(p)$ and we have that $A/C_A(N)$ is a p-group (recall that A is a $\{p, q\}$-group) and in consequence $C_A(N)$ contains some Sylow q-subgroup of A. Hence can write $A = PC_A(N)$. Hence N is in fact a minimal normal subgroup of P.

On one hand, it follows from the previous fact that N is central in P and, since $A = PC_A(N)$, N must be contained in the center of A. This implies that N is a central chief factor of A.

On the other hand, recall that $A/\Phi(A)$ is an $\mathcal{F}$-critical group with trivial Frattini subgroup. Since $\Phi(A) \cap A^{\mathcal{F}} = 1$ and $A^{\mathcal{F}}\Phi(A)/\Phi(A) = (A/\Phi(A))^{\mathcal{F}}$, it follows by Lemma 2 that $A^{\mathcal{F}}$ is abelian. But the equality $A^{\mathcal{N}} = A^{\mathcal{F}} \times N$ yields that $A^{\mathcal{N}}$ is abelian as well and so $A^{\mathcal{N}}$ is complemented in A by a Carter subgroup of A ([4; IV, 5.18]). We can conclude that there exists a Carter subgroup C of A such that $A = A^{\mathcal{N}}C$ with $A^{\mathcal{N}} \cap C = 1$. Now N is central in A. Hence $N \leq N_G(C) = C$. Consequently $N \leq A^{\mathcal{N}} \cap C = 1$, a contradiction. ∎

THEOREM C: *Let $\mathcal{F}$ be a subgroup-closed saturated formation of soluble groups containing the class $\mathcal{N}$ of nilpotent groups. The following statements are equivalent:*

1. *Every soluble group in* $\mathrm{Crit}_S(\mathcal{F})$ *is a Schmidt group.*
2. $G^{\mathcal{F}} = \langle A^{\mathcal{N}} \colon A \leq G; A \in \mathcal{B}_{\mathcal{F}} \rangle$ *for every group* G.

Proof: Let us firstly assume that $\mathrm{Crit}_S(\mathcal{F})$ is contained in $\mathrm{Crit}_S(\mathcal{N})$.

Applying Theorem A, we have that $G^{\mathcal{F}} = \langle A^{\mathcal{F}} \colon A \leq G; A \in \mathcal{B}_{\mathcal{F}} \rangle$. But by hypotheses every soluble group in $\mathrm{Crit}_S(\mathcal{F})$ is a Schmidt group, and thus applying Theorem B, $A^{\mathcal{F}} = A^{\mathcal{N}}$ for every subgroup A of G in $\mathcal{B}_{\mathcal{F}}$. Consequently $G^{\mathcal{F}} = \langle A^{\mathcal{F}} \colon A \leq G; A \in \mathcal{B}_{\mathcal{F}} \rangle = \langle A^{\mathcal{N}} \colon A \leq G; A \in \mathcal{B}_{\mathcal{F}} \rangle$ as we wanted to show.

Therefore, only the sufficiency of the condition (2) is in doubt. To prove that every soluble group in $\mathrm{Crit}_S(\mathcal{F})$ is a Schmidt group, we shall use the characterization given in Theorem 2. Note that a soluble group in $\mathrm{Crit}_S(\mathcal{F})$ cannot be a cyclic group of prime order since $\mathcal{N}$ is contained in $\mathcal{F}$. Write $\mathcal{F} = LF(F)$, where F denotes the canonical local definition of $\mathcal{F}$ (see [4; IV, 3] for details). Moreover, $\mathrm{char}\,\mathcal{F}$ is the set of all primes since $\mathcal{N} \subseteq \mathcal{F}$. Consider any prime p. We shall be done if we prove that $F(p) = \mathcal{S}_{\pi(p)} \cap \mathcal{F}$, where $\pi(p) = \mathrm{char}\,F(p)$.

Since $\mathcal{F}$ is subgroup-closed, applying [4; IV, 3.16] we obtain that $F(p)$ is subgroup-closed as well. Since F is integrated, we have that $F(p) \subseteq \mathcal{F} = \mathcal{S} \cap \mathcal{F}$. Moreover, if given a prime q there exists any q-element in a group belonging to $F(p)$, then a cyclic group of order q belongs to $F(p)$ as well since $F(p)$ is subgroup-closed. Therefore $q \in \mathrm{char}\,F(p)$ and consequently $F(p)$ is contained in $\mathcal{S}_{\pi(p)} \cap \mathcal{F}$. Assume that $\mathcal{S}_{\pi(p)} \cap \mathcal{F} \neq F(p)$ and take a group G in $(\mathcal{S}_{\pi(p)} \cap \mathcal{F}) \setminus F(p)$ of minimal order.

It is clear that $1 \neq \mathrm{Soc}(G)$ is the unique minimal normal subgroup of G, since $F(p)$ is a formation and G is of minimal order. Moreover, $\mathrm{Soc}(G)$ cannot be a p-group since, being F full, it holds that $F(p) = \mathcal{S}_pF(p)$. Note that, in fact, $O_p(G) = 1$.

By [4; B.10.7 and B.10.9], there exists an irreducible and faithful G-module V over $GF(p)$, the finite field of p elements. Consider now the corresponding semidirect product $X = [V]G$. Note that if $X \in \mathcal{F}$, then $X/C_X(V) \in F(p)$ and thus $X/V \in F(p)$. This is impossible since $G \cong X/V$. Therefore $X \notin \mathcal{F}$ and X is in fact an $\mathcal{F}$-critical group.

We are ready at this point to reach our final contradiction. Since $X^{\mathcal{F}} = \langle A^{\mathcal{N}}: A \leq X; A \in \mathcal{B}_{\mathcal{F}} \rangle$, and $X \in \mathcal{B}_{\mathcal{F}}$, we have that $X^{\mathcal{F}} = X^{\mathcal{N}} = V$. Therefore $G \cong X/V = X/X^{\mathcal{N}}$ is nilpotent, a contradiction. ∎

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